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ON THE HOMOGENEOUS CONE $z^2 = 53x^2 + y^2$

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Abstract:

The homogeneous ternary quadratic equation given by $z^2 = 53x^2 + y^2$ is analysed for its non-zero distinct integer solution through different methods. A few interesting properties between the solution are presented. Also, formulae for generating sequence of integer solutions based on the given solutions are presented.

Keywords: Ternary quadratic, Integer solutions, Homogeneous cone.

Notation:

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

Introduction:

It is well known that the quadratic Diophantine equations with three unknowns (homogeneous or non-homogeneous) are rich in variety [1, 2]. In particular, the ternary quadratic Diophantine equations of the form $z^2 = Dx^2 + y^2$ are analysed for values of D=29, 41, 43, 47, 61, 67 in [3-8]. In this communication, the homogeneous ternary quadratic Diophantine equation given by $z^2 = 53x^2 + y^2$ is analysed for its non-zero distinct integer solution through different methods. A few interesting properties between the solutions are presented. Also, formulas for generating sequence of integer solutions based on the given solutions are presented.

METHODS OF ANALYSIS

The ternary quadratic equation to be solved for its integer solutions is

$$z^2 = 53x^2 + y^2 \tag{1}$$

We present below different methods of solving (1):

Method: 1

(1) Is written in the form of ratio as

$$\frac{z+y}{53} = \frac{x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{2}$$

which is equivalent to the system of double equations

$$53\alpha x - \beta y - \beta z = 0$$
$$\beta x + \alpha y - \alpha z = 0$$

Applying the method of cross-multiplication to the above system of equations, one obtains

$$x = x(\alpha, \beta) = 2\alpha\beta$$
$$y = y(\alpha, \beta) = 53\alpha^2 - \beta^2$$
$$z = z(\alpha, \beta) = 53\alpha^2 + \beta^2$$

Which satisfy (1)

Properties:

- $y(\alpha,1) - t_{108,\alpha} + 1 = 0$
- $2[y(\alpha,1) + z(\alpha,1)] = 53x^3(\alpha,1)$
- $z(\alpha,1) - t_{108,\alpha} - 1 = 0$
- $y(\alpha,1) + z(\alpha,1) = 2t_{108,\alpha}$

Note: 1

It is observed that (1) may also be represented as below:

$$\frac{z+y}{x} = \frac{53x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0$$

In this case, the corresponding solutions to (1) are given as:

$$x = 2\alpha\beta, \quad y = \alpha^2 - 53\beta^2, \quad z = \alpha^2 + 53\beta^2$$

Method: 2

(1) Is written as the system of double equation in Table 1 as follows:

Table: 1 System of Double Equations

System	**I**	**II**	**III**
$z + y =$	x^2	$53x^2$	$53x$
$z - y =$	53	1	x

Solving each of the above system of double equations, the value of $x, y \& z$ satisfying (1) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

$$x = 2k + 1, \quad y = 2k^2 + 2k - 26, \quad z = 2k^2 + 2k + 27$$

Solutions for system: II

$$x = 2k + 1, \quad y = 106k^2 + 106k + 26, \quad z = 106k^2 + 106k + 27$$

Solution for system: III

$$x = k, \quad y = 26k, \quad z = 27k$$

Method: 3

Let $z = y + k, k \neq 0$

$$\therefore (1) \Rightarrow 2ky = 53x^2 - k^2 \qquad (3)$$

Assume

$$x = 2k\alpha + k = k(2\alpha + 1) \qquad (4)$$

$$\therefore\ y = 53(2k\alpha^2 + 2k\alpha) + 26k \quad (5)$$

In view of (3)

$$z = 53(2k\alpha^2 + 2k\alpha) + 27k \quad (6)$$

Note that (4), (5), (6) satisfy (1)

Method: 4

(1) Is written as

$$y^2 + 53x^2 = z^2 = z^2 * 1 \quad (7)$$

Assume z as

$$z = a^2 + 53b^2 \quad (8)$$

Write 1 as

$$1 = \frac{\left(22 + 3i\sqrt{53}\right)\left(22 - 3i\sqrt{53}\right)}{31^2} \quad (9)$$

Using (8) & (9) in (7) and employing the method of factorization, consider

$$\left(y + i\sqrt{53}x\right) = \left(a + i\sqrt{53}b\right)^2 \cdot \frac{22 + 3i\sqrt{53}}{31}$$

Equating real & imaginary parts, it is seen that

$$\left.\begin{aligned} x &= 3\left(a^2 - 53b^2\right) + 44ab \\ y &= 22\left[a^2 - 53b^2 - 318ab\right] \end{aligned}\right\} \quad (10)$$

Since our interest is to find the integer solutions, replacing a by 31A & b by 31B in (8) & (10), the corresponding integer solutions to (1) are given by

$$\begin{aligned} x &= x(A,B) = 31\left[3A^2 - 159B^2 + 44AB\right] \\ y &= y(A,B) = 31\left[22A^2 - 1166B^2 - 318AB\right] \\ z &= z(A,B) = 31^2\left[A^2 + 53B^2\right] \end{aligned}$$

Properties:

- $\frac{5x(A,1)}{31} - t_{32,A} = 234A - 795$
- $\frac{5x(A,1)}{31} - t_{32,A} \equiv 141 (\mathrm{mod}\ 234)$

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- $\frac{5y(A,1)}{31} - t_{222,A} = -[1481A + 5830]$
- $\frac{5y(A,1)}{31} - t_{222,A} \equiv 94 \pmod{1481}$

Note: 2

In addition to (5), 1 may also be represented as follows

$$1 = \frac{\left[\left(53r^2 - s^2\right) + i\sqrt{53}\cdot 2rs\right]\left[\left(53r^2 - s^2\right) - i\sqrt{53}\cdot 2rs\right]}{\left(53r^2 + s^2\right)^2}$$

For the above choice, the corresponding values of x, y, z satisfying (1) are given below:

$$x = 53r \quad + s^2\left[2rs\left(A^2 - 53B^2\right) + 2AB\left(53r^2 - s^2\right)\right]$$
$$y = \left(53r^2 + s^2\right)\left[\left(53r^2 - s^2\right)\left(A^2 - 53B^2\right) - 212ABrs\right]$$
$$z = \left(53r^2 + s^2\right)^2\left(A^2 + 53B^2\right)$$

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let $(x_0, y_{0,} z_0)$ be any given solution to (1)

Formula: 1

Let $(x_{1,} y_1, z_1)$ given by

$$x_1 = 3x_0, \qquad y_1 = 3y_0 + h, \qquad z_1 = 2h - 3z_0 \tag{11}$$

Be the 2^{nd} solution to (1). Using (11) in (1) and simplifying, one obtains

$$h = 2y_0 + 4z_0$$

In view of (11), the values of y_1 and z_1 are written in the matrix form as

$$(y_1, z_1)^t = M(y_0, z_0)^t$$

Where

M= $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ and t is the transpose

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The repetition of the above proses leads to the n^{th} solutions y_n, z_n given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 1, \quad \beta = 9$$

We know that

$$M^n = \frac{a^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I), \quad I = 2 \times 2 \text{ Identity matrix}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 3^n x_0$$
$$y_n = \left(\frac{9^n + 1}{2}\right) y_0 + \left(\frac{9^n - 1}{2}\right) z_0$$
$$z_n = \left(\frac{9^n - 1}{2}\right) y_0 + \left(\frac{9^n + 1}{2}\right) z_0$$

Formula: 2

Let (x_1, y_1, z_1) given by

$$x_1 = h - 54x_0, \quad y_1 = h - 54y_0, \quad z_1 = 54z_0 \qquad (12)$$

Be the 2^{nd} solution to (1). Using (12) in (1) and simplifying, one obtains

$$h = 106x_0 + 2y_0$$

In view of (12), the values of y_1 and z_1 is written in the matrix form as

$$(x_1, y_1)^t = M (x_0, y_0)^t$$

Where

$$M = \begin{pmatrix} 52 & 2 \\ 106 & -52 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_o, y_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 54, \beta = -54$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 54^{n-1}\left(53 + (-1)^n\right)x_0 + 54^{n-1}\left(1 - (-1)^n\right)y_0$$
$$y_n = 54^{n-1} \cdot 53\left(1 - (-1)^n\right)x_0 + 54^{n-1}\left(1 - 53(-1)^n\right)y_0$$
$$z_n = 54^n z_o$$

Formula: 3

Let $(x_1, y_1 z_1)$ given by

$$x_1 = h - 2x_0, \quad y_1 = 2y_0, \quad z_1 = 2z_0 + 7h \tag{13}$$

Be the 2^{nd} solution to (1). Using (13) in (1) and simplifying, one obtains

$$h = 53x_0 + 7z_0$$

In view of (13), the values of x_1 and z_1 is written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

Where

$$M = \begin{pmatrix} 51 & 7 \\ 371 & 51 \end{pmatrix}$$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 51 + 7\sqrt{53}, \qquad \beta = 51 - 7\sqrt{53}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \left(\frac{\alpha^n + \beta^n}{2}\right)x_0 + \frac{1}{\sqrt{53}}\left(\frac{\alpha^n - \beta^n}{2}\right)z_0$$
$$y_n = 2^n y_0$$
$$z_n = \sqrt{53}\left(\frac{\alpha^n - \beta^n}{2}\right)x_0 + \left(\frac{\alpha^n + \beta^n}{2}\right)z_0$$

Conclusion:

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic Diophantine equation $z^2 = 53x^2 + y^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

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